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Multi-dimensional maps with infinite invariant measures and countable state sofic shifts

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Abstract

We study multi-dimensional maps on bounded domains of $\mathbf{R}^d$ satisfying the finite range structure (FRS) condition, which leads us to countable state sofic systems. Such maps admit σ-finite ergodic invariant measures equivalent to Lebesgue measures under the local Renyi condition . In this paper we show that several ergodic properties still hold even if such invariant measures are infinite . We also investigate on the validity of Rohlin's entropy formula and Variational principle for the entropy.

1 Introduction

We study piecewise invertible maps with finite range structure (FRS) whose symbolic dynamics are countable state sofic shifts. There are examples which do not satisfy the Markov condition but satisfy the FRS condition (see section 11). More specifically, assume the following conditions:

1. $T : X \to X$ is a map on a bounded domain X of $\mathbf{R}^d$.
2. $Q = \{X_a\}_{a\in I}$ is a generating countable partition of X, consists of measurable connected subsets with piecewise smooth boundaries.
3. For each $X_a, T|_{X_a} : X_a \to TX_a$ is a homeomorphism.
4. Denote $X_{a_1} \cap T^{-1}X_{a_2} \cap \ldots \cap T^{-(n-1)}X_{a_n}$ by $X_{a_1\ldots a_n}$ if its interior is not empty. (Or its interior has positive Lebesgue measure. It is also possible to work with this measure theoretical definition.) Put $\mathcal{U} = \{T^n X_{a_1\ldots a_n} : \forall X_{a_1\ldots a_n}, \forall n > 0\}$. Then $\mathcal{U}$ consists of only finitely many subsets of X with positive Lebesgue measures. (FRS condition.)

We call the quadraple $(T, X, Q, \mathcal{U})$ a piecewise invertible system with FRS. Even if these systems do not satisfy the Markov condition, the condition 4 leads us to nice countable state symbolic dynamics, "sofic systems". Many examples of such maps come from number theory ([8],[9],[11],[18],[22],[23]). If we assume further that T is a piecewise C^1 map so that 3 becomes

3* $T|_{X_a}$ is a C^1-diffeomorphism for all $X_a \in Q$,

we can obtain nice invariant measures under certain regularity conditions. To specify the conditions, we need some notations. Put

$$C(a_1 \dots a_n) = \frac{\sup_{x \in X_{a_1 \dots a_n}} |\det DT^n(x)|}{\inf_{x \in X_{a_1 \dots a_n}} |\det DT^n(x)|}$$

for a cylinder $X_{a_1 \dots a_n}$, and define for given $C > 1, \mathcal{R}(C.T)$ by the set of cylinders $X_{a_1 \dots a_n}$ satisfying $C(a_1 \dots a_n) < C$. $\mathcal{L}^{(n)}$ denotes the family of all cylinders of rank n and $\mathcal{L} = \bigcup_{n=1}^{\infty} \mathcal{L}^{(n)}$. The following condition gives a uniform bounds for build-up of non-linearity under iteration of T.

(Renyi's condition) There exists a constant $C > 1$ such that $\mathcal{R}(C.T) = \mathcal{L}$.

If T satisfies such a bounded distortion property, then the well-known approach by using a Perron-Frobenius operator applies , so that we could obtain an ergodic invariant measure with density bounded away from zero and infinity, and further ergodic properties were established([3],[4],[5],[9],[11], [12],[16],[21],[25],[26]). Even if Renyi's condition does not hold, still we could have nice invariant measures which admit unbounded densities , under the FRS condition and the local Renyi condition in [8]. We can find many examples of such maps in [23]([22]).

Theorem 1.1 *(***The existence of an ergodic invariant measure equivalent to Lebesgue measure***)*

Let $(T, X, Q, \mathcal{U})$ be a piecewise invertible system with FRS satisfying the condition 3. Assume that there exists a constant $C > 1$ suth that $\mathcal{R}(C.T) \neq \emptyset$ and satisfies the following conditions:*

(1-1) $X_{b_1 \dots b_k a_1 \dots a_n} \in \mathcal{R}(C.T)$ if $X_{a_1 \dots a_n} \in \mathcal{R}(C.T)$. (The local Renyi condition.)

(1-2) Each $U \in \mathcal{U}$ contains a cylinder $X_{a_1 \dots a_s}$ satisfying $X_{a_s} \in \mathcal{R}(C.T)$ and $T^s X_{a_1 \dots a_s} = X$. (The transitivity condition.)

(1-3) Define $\mathcal{D}_n = \{X_{a_1 \dots a_n} : X_{a_1 \dots a_i} \notin \mathcal{R}(C.T)$ for $1 \leq i \leq n\}$ and put $D_n = \bigcup_{X_{a_1 \dots a_n} \in \mathcal{D}_n} X_{a_1 \dots a_n}$. (For $n = 0$ we put $D_0 = X$.) Then $\lim_{n \to \infty} \lambda(D_n) = 0$, where λ is the normalized Lebesgue measure of X.

Under the above conditions, the $(T, X, Q, \mathcal{U})$ admits an ergodic σ-finite invariant measure μ equiovalent to λ . If we assume

*(1-3)** $\sum_{n=0}^{\infty} \lambda(D_n) < \infty$

instead of (1-3),then μ is finite.

Remark

(1) The condition (1-1) seems to imply a finite memory of distortion of T and is called the *local Renyi condition.*

(2) The condition (1-3) with (1-2) plays an important role in constructing a jump transformation on a full measure set ,which satisfies Renyi's condition so that admits an ergodic invariant measure ν equivalent to λ with a bounded density. The invariant measure μ is given by using ν as (2-1) in section 2.

(3) Even if μ is finite, the invariant density of μ is not necessarily bounded.

In previous papers ([22],[23]), we obtained under the condition (1-3)*, exactness, a characterization of singular points of the invariant density of μ, and showed a sufficient condition for the validity of Rohlin's entropy formula. In this paper, we will investigate these facts under the condition (1-3), i.e., in case μ is infinite in section 4,7, and prove further results, "rationally ergodicity"(in section 5),"wandering rates " (in section 6). In previous works, Ergodic theory for the case of infinite invariant measures were developed by several people. For example, inner functions of upper half plane ([1]), and more generally piecewise-C^2 Bernoulli maps on $[0,1]$ with indifferent fixed points ([19],[20]) are good examples of one-dimensional maps preserving infinite invariant measures. For multi-dimensional maps , Markov fibred systems based on [14] have been studied in [2] with application to parabolic rational maps on the Riemann sphere. These systems are examples of piecewise invertible systems with FRS satisfying the Markov condition. Our results in section 4, 5, 6, and 7 are mostly obtained by generalizing the idea of proofs in [20] and [2]. In section 8, we investigate on "variational principle for the entropy" under the condition (1-3)* again, applying the results of [25] in which maps satisfy the bounded distortion property (Renyi's condition), even if we do not have such a property . In section 9, we also discuss on the relation between FRS and countable state sofic shift. We give some examples of our results in section 10, which occur from number theory and suggest that countable state sofic shifts can be products of finite state sofic shifts and countable Bernoulli shifts.

2 Conservativity and Jump transformations

We assume all of conditions in the previous section, throughtout this paper, except Section 10,i.e, $(T, X, Q = \{X_a\}_{a\in I}, \mathcal{U})$ is a piecewise invertible system with FRS satisfying 3*, (1-1),(1-2), and (1-3). We first prepare some notation.

Let $(X, \mathcal{F}, \lambda)$ be the normalized Lebesgue space. $\mathcal{L}^n$ denotes the set of all cylinders with respect to T and put $\mathcal{L} = \bigcup_{n\geq 1} \mathcal{L}^n$. For $n \geq 1$, put

$$\mathcal{B}_n = \{X_{a_1\ldots a_n} \in \mathcal{L}^n : X_{a_1\ldots a_{n-1}} \in \mathcal{D}_{n-1}, X_{a_1\ldots a_n} \in \mathcal{R}(C.T)\}$$

and denote $B_n = \bigcup_{X_{a_1\ldots a_n}\in\mathcal{B}_n} X_{a_1\ldots a_n}$. It follows from (1-3) that $\bigcup_{n=1}^{\infty} B_n = X(\lambda \bmod 0)$ (see [8]). Define a map $T^* : \bigcup_{n=1}^{\infty} B_n \to X$ by $T^*x = T^j x$ for $x \in B_j$. We call the T^* a *jump transformation over $R(C.T)$*. By restricting T^* on $X \backslash \bigcup_{m=0}^{\infty} T^{*-m}(\bigcap_{n\geq 0} D_n)$, we have a transformation of X^*, and as $X^* = X(\lambda \bmod 0)$ we use the same notation T^* for this restriction on X^*. Define $I^* = \bigcup_{n=1}^{\infty}\{(a_1 \ldots a_n) \in I^n : X_{a_1\ldots a_n} \in \mathcal{B}_n\}$ and $Q^* = \{X_\alpha\}_{\alpha\in I^*}$. For $\alpha \in I^*$ we denote the length of the sequence α by $|\alpha|$. $\mathcal{L}^{*(n)}$ and $\mathcal{L}^*$ are defined as well as above, with respect to T^*. Put $\mathcal{U}^* = \{T^{*n}X_{\alpha_1\ldots\alpha_n} : n > 0, X_{\alpha_1\ldots\alpha_n} \in \mathcal{L}^{*(n)}\}$. Then $(T^*, X^*, Q^* = \{X_\alpha\}_{\alpha\in I^*}, \mathcal{U}^*)$ is a piecewise invertible system with FRS satisfying the Renyi's condition for the $C > 1$, so T^* admits a finite ergodic invariant measure ν equivalent to λ with density bounded away from zero and infinity ($G^{-1} \leq d\nu/d\lambda \leq G$). μ is given by using this T^*-invariant measure ν as follows:

(2-1) $\mu(E) = \sum_{n=0}^{\infty} \nu(T^{-n}E \cap D_n)$ $(\forall E \in \mathcal{F})$. (Cf [8].)

From the above formula, we can also find the formula of the invariant density of μ:

(2-2)
$$\frac{d\mu}{d\lambda}(x) = \sum_{n=0}^{\infty} \sum_{X_{d(n)}\in\mathcal{D}_n} |\det D\psi_{d(n)}(x)| \frac{d\nu}{d\lambda}(\psi_{d(n)}(x)) I_{T^n X_{d(n)}}(x)$$

for a.e. $x \in X$, (where $d(n)$ stands for $(d_1 \ldots d_n)$). As ν is finite, T^* is conservative with respect to ν and so that with respect to λ. This fact leads us to the following:

Theorem 2.1 *T is conservative.*

3 Induced transformations and the uniqueness of μ

We call the constant $C(> 1)$ satisfying (1-1) of Theorem 1.1, the *local Renyi constant* for T, and we say that T satisfies the *local Renyi condition* if T admits such a local Renyi constant. The formula (2-1) in section 2 seems to suggest a dependence of μ on C, because the jump transformation itself depends on the local Renyi constant. As the μ we obtained from (2-1) is usually ergodic, equivalent to λ, so under the condition (1-3)* we can say that the existence of μ is unique. Even if (1-3)* does not hold, the condition (1-3) under which μ can be infinite is enough to obtain such a uniqueness of the existence of μ. We can prove this by two ways, one of these uses the induced transformation and the other uses the "Chacon-Ornstein-Silva-Thieullen's ratio ergodic theorem".([15].)

Theorem 3.1 *Asuume that T satisfies the local Renyi condition. Let $C > C' > 1$ be the local Renyi constants for T. Let μ^C and $\mu^{C'}$ be T invariant measures which are given by the formula (2-1),respectively. Then for any $A \in \mathcal{F}$ satisfying*

$$0 < \mu^{C'}(A) < \infty, \textit{we have } \mu^C|_A = \mu^{C'}|_A.$$

Remark A $\mu^C \leq G^2\mu^{C'}$ for $C' < C$.

In fact, as $\mathcal{R}(C'.T) \subseteq \mathcal{R}(C.T)$ and so $D_n^C \subseteq D_n^{C'}(\forall n \geq 0)$,

$$G\mu^{C'}(E) \geq \sum_{n=0}^{\infty} \lambda(T^{-n}E \cap D_n^{C'})$$

$$\geq \sum_{n=0}^{\infty} \lambda(T^{-n}E \cap D_n^{C}) \geq G^{-1}\mu^C(E)$$

for $\forall E \in \mathcal{F}$.

Corollary 3.1 *For $\forall A \in \mathcal{F}$ with $0 < \mu^{C'}(A) < \infty$, $\mu^C(A) = \mu^{C'}(A)$.*

As T is conservative with respect to λ, we can define the induced transformation T_A over $A \in \mathcal{F}$ with positive finite measure. Let us define

$$A_1 = A \cap T^{-1}A, \quad A_k = A \cap (\bigcap_{j=1}^{k-1} T^{-j}A^c) \cap T^{-k}A \quad (k \geq 2)$$

inductively. Put $F_0 = A$ and we define $F_k = A \cap (\bigcap_{j=1}^{k} T^{-j}A^c)$ for $k \geq 1$. Then $A = (\bigcup_{n=1}^{l} A_n) \cup F_l$ is a disjoint union for each $l \geq 1$, and it follows from the conservativity of T that $\bigcup_{n=1}^{\infty} A_n = A$ (λ mod 0). So T_A is defined on $\bigcup_{n=1}^{\infty} A_n$ and $T_A x = T^n x$ for $x \in A_n$.

Lemma 3.1 *$\mu|_A$ gives T_A-invariant measure which is finite, ergodic and equivalent to $\lambda|_A$.*

Lemma 3.2 *Let $0 < \lambda(A) < \infty$ and let ν_A be a finite T_A-invariant measure. Then the following (Kac's) formula gives T-invariant measure; i.e., define*

$$\mu_{\nu_A}(E) = \sum_{k=0}^{\infty} \nu_A(F_k \cap T^{-k}E),$$

where $F_0 = A$. Then μ_{ν_A} is a T-invariant measure.

Remark B The finiteness of μ_{ν_A} is determined by the finiteness of $\sum_{k=1}^{\infty} \nu_A(F_k)$. Note that $\bigcup_{k=1}^{\infty} F_k = A(\lambda \bmod 0)$ and so μ_{ν_A} is σ-finite. Assume that $\nu_A << \lambda|_A$. Then even if $\sum_{k=1}^{\infty} \nu_A(F_k) = \infty$, the conservativity of T with respect to λ allows us to have:

$$\lim_{k \to \infty} \nu_A(F_k) = \bigcap_{k=0}^{\infty} \nu_A(F_k) = 0.$$

Lemma 3.3 *If $\bigcup_{j=0}^{\infty} T^{-j}A = X(\lambda \bmod 0)$, then $\mu_{\mu|_A} = \mu$.*

Remark C The conservativity and the ergodicity of T with respect to λ allows us to have $\bigcup_{k=0}^{\infty} T^{-k}A = X(\lambda \bmod 0)$.

4 Exactness

As we mentioned in section 1, we have already obtained exactness of (T,μ) under the condition (1-3)*. In this section, we only assume (1-3) which admits an infinite invariant measure.

Theorem 4.1 *Under all conditions of Theorem 1.1, T is an exact endomorphism.*

Lemma 4.1 *Let $(T, X, Q, \mathcal{U})$ be a piecewise invertible system with FRS and admit a constant $C > 1$ such that $\mathcal{R}(C.T) = \mathcal{L}$, i.e., T satisfy Renyi's condition. Then for any measurable set A and for any cylinder $X_{a_1\ldots a_k} \in \mathcal{L}$, we have*

$$C^{-1}\lambda(T^k X_{a(k)} \cap A) \leq \frac{\lambda(X_{a(k)} \cap T^{-k}A)}{\lambda(X_{a(k)})} \leq C(\min\{\lambda(U) : U \in \mathcal{U}\})^{-1}\lambda(T^k X_{a(k)} \cap A).$$

(Here $a(k)$ stands for $a_1 \ldots a_k$.)

5 Rationally ergodicity

Definition

A conservative ergodic measure preserving transformation on a σ-finite measure space $(X, \mathcal{F}, \mu)$ is called *rationally ergodic*, if there exists a set $A \in \mathcal{F}$ of positive measure such that

(5-1)

$$\sup_{n\geq 1}\{\int_A (\textstyle\sum_{k=0}^{n-1} 1_A T^k / a_n(A))^2 d\mu(x)\} < \infty,$$

where $a_n(A) = \sum_{k=0}^{n-1}\mu(A \cap T^{-k}A)$. This condition (5-1) implies that the following ratio limiting theorem holds for all $A_1, A_2, C_1, C_2 \in A \cap \mathcal{F}$ of positive measure

(5-2)

$$\lim_{n\to\infty}\frac{\sum_{k=0}^{n-1}\mu(A_1 \cap T^{-k}C_1)/\mu(A_1)}{\sum_{k=0}^{n-1}\mu(A_2 \cap T^{-k}C_2)/\mu(A_2)} = \mu(C_1)/\mu(C_2).$$

([20]).

Remark D We have already known the following fact from the "Chacon-Ornstein's ratio ergodic theorem " : μ a.e. $x \in X$

$$\exists \lim_{n\to\infty}\frac{\sum_{k=0}^{n-1} 1_{C_1}(T^k x)}{\sum_{k=0}^{n-1} 1_{C_2}(T^k x)} = \mu(C_1)/\mu(C_2).$$

Under some additional conditions, we have further result.

(5-3) there exists a sequence $\{M_n\}_{n\geq 1}$ such that for each $n \geq 1$

$$C(a_1 \ldots a_n) \leq M_n \text{ for } \forall X_{a_1 \ldots a_n} \in \mathcal{L}^{(n)}.$$

Remark E Under (5-3) we can obtain a monotone increasing sequence $\{M_n'\}_{n\geq 1}$ such that for each $n \geq 1$

$$C(a_1 \ldots a_n) \leq M_n' \text{ for } \forall X_{a_1 \ldots a_k} \in \mathcal{L}^{(k)} \text{ and } \forall k \in \{1, 2, \ldots n\}.$$

In fact, it is enough to put $M_n' = \max\{M_{n-1}', M_n\}(M_1' = M_1)$ inductively.

For $m \geq 1$, put

$$W_m = \sum_{n=0}^{\infty} \sum_{X_{d(n)} \in \mathcal{D}_n} \sup_{x \in T^n X_{d(n)} \cap (\bigcup_{k=1}^{m} B_k)} |\det D\psi_{d(n)}(x)|.$$

Then

(5-4)

$$W_m < \infty \text{ for } \forall m \geq 1.$$

Theorem 5.1 *Suppose that all assumptions of Theorem1.1 are satisfied. Assume further (5-3) and (5-4). Then T is rationally ergodic. In fact for each $m \geq 1, \bigcup_{k=1}^{m} B_k$ satisfies (5-1).*

Remark F The quantity W_m can be checked explicitly for many number theoretical transformations ([22],[23]).

Lemma 5.1 *(5-4) implies that*

$$\frac{d\mu}{d\lambda}(x) \leq W_m < \infty \text{ for a.e. } x \in \bigcup_{k=1}^{m} B_k.$$

6 On wandering rates

In this section, under the assumption $\sum_{n=0}^{\infty} \lambda(D_n) = \infty$ with (1-3), we will prove the results on wandering rates which are a generalization of Theorem 4.1 of [2] (Cf.[20]).

Let $A \in \mathcal{F}$ and put $A_0 = A$. We define A_k for $k \geq 1$ inductively by

$$A_k = T^{-k} A \backslash (\bigcup_{j=0}^{k-1} T^{-j} A).$$

The *wandering rate* of A is defined by

$$L_A(n) = \mu(\bigcup_{l=0}^{n} T^{-l} A) = \sum_{k=0}^{n} \mu(A_k).$$

Let $\mathcal{F}_0$ denotes the ring generated by $\mathcal{R}(C.T)$.

Theorem 6.1 *Under all conditions of Theorem 1.1, assume further that* $\sum_{n=0}^{\infty}\lambda(D_n) = \infty$ *and (5-4) holds. If* $A \in \mathcal{F}$ *be a finite union of sets in* $\mathcal{R}(C.T)$, *then* $L_A(n) \sim L_B(n)$ *as* $n \to \infty$, *for* $\forall B \in \mathcal{F} \cap A, \lambda(B) > 0$.

Corollary 6.1 *There exists an increasing sequence* $L(n)$ *such that*

$$L_B(n) \sim L(n)$$

as $n \to \infty$ *for* $\forall A \in \mathcal{F}_0, B \in A \cap \mathcal{F}, \lambda(B) > 0$.

Lemma 6.1 *Let* $X_{a(k)} \in \mathcal{R}(C.T), B \in \mathcal{F} \cap X_{a(k)}$, *and put*

$$L = (\max\{\lambda(U_i) : 0 \le i \le N\}/\min\{\lambda(U_i) : 0 \le i \le N\})^2.$$

Then for $\forall n > 0$ *and* $\forall X_{b(n)} \in \mathcal{L}^{(n)}$ *with* $\lambda(X_{b(n)} \cap T^{-n}B) > 0$, *there exists a constant* $\gamma \in [C^{-4}L^{-1}, C^4 L]$ *satisfying :*

$$\frac{\lambda(X_{b(n)} \cap T^{-n}B)}{\lambda(X_{b(n)a(k)})} = \gamma \frac{\lambda(B)}{\lambda(X_{a(k)})}.$$

(Cf.Lemma 2.3 in [2].)

7 Rohlin's entropy formula

We have already obtained a sufficient condition for the validity of Rohlin's formula under the condition (1-3)* which guarantees the finiteness of μ ([23]). Here we will discuss on the validity of the Rohlin's formula in case μ is infinite.

(7-1) There exists a cylinder $X_a \in \mathcal{B}_1$ such that

$$\log|\det DT_{X_a}(x)| \in L^1(X_a, \lambda|_{X_a}),$$

where T_{X_a} denotes the induced transformation of T over X_a.

Theorem 7.1 *Under the condition (7-1),*

$$h(\mu, T) = \int_X \log|\det DT(x)|d\mu(x) < \infty.$$

Remark H For $A \in \mathcal{F}$ with $0 < \mu(A) < \infty$, the number $\mu(A)h(\frac{\mu|_A}{\mu(A)}, T_A)$ is independent of A, so that the number is defined as entropy of T with respect to μ.

Remark I As the induced system $(T_{X_a}, \mu|_{X_a})$ satisfies the Renyi's condition, $\mu|_{X_a}$ is a finite invariant measure with density bounded away from zero and infinity, so that (7-1) is enough to obtain the following formula:

$$h(T_{X_a}, \mu|_{X_a}) = \int_{X_a} \log|\det DT_{X_a}(x)|d(\mu|_{X_a})(x)$$

(see [13],[23]).

Remark J Under the condition (7-1), we have for $A = X_a$ the following :

$$h(\mu, T) = \mu(A)h(\frac{\mu|_A}{\mu(A)}, T_A) = \int_A \log|\det DT_A(x)|d(\mu|_A)(x) < \infty.$$

Lemma 7.1 *For $A = X_a$*

$$\int_A \log|\det DT_A(x)|d(\mu|_A)(x) = \int_X \log|\det DT(x)|d\mu(x) < \infty.$$

8 Variational principle for the entropy

In [25], the existence and ergodic properties of maps which expand distances satisfying the bounded distortion property (Renyi's condition) were established along the line of well-known theory. The key to the proof are to define a Perron-Frobenius operator acting on a suitable space of measurable functions and to show the convergence theorem for the powers of the operator. As mentiond in Introduction, we can not use Renyi's condition so that we can not use any such a operator. However we have already succeeded to obtain nice invariant measures([8]) and Rohlin's entropy formula in case μ is finite([23]).

First we summarize the notations . Let $\overline{X}$ be a compact metric space and X be an open dense subset of $\overline{X}$. $C(\overline{X})$ denotes the Banach space of real-valued continuous functions on $\overline{X}$, and $C(X)$ denotes the space of continuous functions on X.$\mathcal{M}(\overline{X})$ denotes the collection of all probability measures on the σ-algebra of Borel sets of $\overline{X}$. If $\mu \in \mathcal{M}(\overline{X})$ and $f \in C(\overline{X})$ $\mu(f)$ denotes the integral of f with respect to μ. Let X_0 be an open dense subset of X and suppose $T : X_0 \to X$ is a continuous map of X_0 onto X such that $\{T^{-1}x\}$ is at most countable for each $x \in X$.

Remark K A piecewise invertible system $(T, X, Q, \mathcal{U})$ satisfies the above conditions. It suffices to see that $X_0 = X \setminus \bigcup_{a \in I} \partial X_a$ and $\lambda(X_0) = \lambda(X)$.

Let $\mathcal{M}(X)$ be the collection of all probability measures defined on the Borel subsets $\mathcal{F}$ of X and $\mathcal{M}(X, T)$ denotes the collection of all T-invariant ones. For $m \in \mathcal{M}(X)$ and for a subalgebra $\mathcal{F}'$ of $\mathcal{F}$, $E_m(f|\mathcal{F}')$ denotes the conditional expectation of f with respect to $\mathcal{F}'$ and $I_m(\mathcal{F}|\mathcal{F}')$ denotes the conditional information of $\mathcal{F}$ with respect to $\mathcal{F}'$.

Definition

We say that $\mu \in \mathcal{M}(X, T)$ is an *equilibrium state* for $\varphi \in C(X_0)$ if

(8-1)

$$\mu(I_\mu(\mathcal{F}|T^{-1}\mathcal{F}) + \varphi) \geq m(I_m(\mathcal{F}|T^{-1}\mathcal{F}) + \varphi)$$

for $\forall m \in \mathcal{M}(X, T)$.

Let $h(x)$ be the invariant density of μ, i.e., $d\mu/d\lambda(x) = h(x)$. If Renyi's condition is valid, then $h(x) \in C(\overline{X})$ so that $h(x)$ is uniformly bounded from above and below. Thus

$$g(x) = \frac{h(x)}{|\det DT(x)| hT(x)}$$

belongs to $C(X_0)$. Without such a regular condition, we can not obtain such a evidence. However we can see the follwing fact:

Lemma 8.1 $\sum_{y\in T^{-1}x} g(y) = 1$.

Let $f \in C(\overline{X})$. Then $\exists F > 0$ such that $f(x) \in [F^{-1}, F]$ on $\overline{X}$, so that from Lemma 8.1 we have

$$\sum_{y\in T^{-1}x} g(y)f(y) \in [F^{-1}, F].$$

Thus $\sum_{y\in T^{-1}x} g(y)f(y)$ is uniformly bounded on X, so that integrable on X. This consideration leads to the following important property of $g(x)$ for our purpose.

Lemma 8.2

$$\int_X \sum_{y\in T^{-1}x} g(y)f(y)d\mu(x) = \int_X f(x)d\mu(x) \ (\forall f \in C(\overline{X}))$$

Lemma 8.3 *μ-a.e. $x \in X$*

$$E_\mu(f|T^{-1}\mathcal{F})(x) = \sum_{y\in T^{-1}Tx} g(y)f(y).$$

Lemma 8.4 $I_\mu(\mathcal{F}|T^{-1}\mathcal{F})(x) = -\log g(x)$.

More generally, we have:

Proposition 8.1 *Let $m \in \mathcal{M}(X,T)$ and let $g_m : X_0 \to \mathbb{R}$ be the function defined a.e. m by*

$$E_m(f|T^{-1}\mathcal{F})(x) = \sum_{y\in T^{-1}Tx} g_m(y)f(y).$$

Then

$$I_m(\mathcal{F}|T^{-1}\mathcal{F}) = -\log g_m$$

and

$$\int_X \{I_m(\mathcal{F}|T^{-1}\mathcal{F}) + \log g(x)\}dm(x) \leq 0.$$

Conbining the above results we can obtain the desired result.

Theorem 8.1 *The T-invariant finite measure μ obtained in Theorem 1.1 is an equilibrium state for $-\log|\det DT(x)|$. If Rohlin's entropy formula holds, then we can restate (8-1) as*

$$h_\mu(T) - \int_X \log|\det DT(X)|d\mu(x) \geq m(I_m(\mathcal{F}|T^{-1}\mathcal{F}) - \log|\det DT(x)|)$$

for $\forall m \in \mathcal{M}(X,T)$.

Remark L We have already obtained a sufficient condition for the validity of Rohlin's entropy formula in [23].

9 FRS and countable state sofic shifts

As mentioned before, the well-known way to show the existence of nice invariant measures by using Perron-Frobenius operator do not rely on any information from their symbolic dynamics, but rely on Renyi's condition and the Markov condition. On the other hand, we can find easily examples of multi-dimensional piecewise smooth maps which do not satisfy both of these conditions but admit nice invariant measures. Such maps typically satisfy Renyi's condition "locally" (i.e., the local Renyi condition) and the FRS condition. We can see that the FRS condition plays an important role in showing the existence and further metrical properties of such invariant measures([8],[22],[23]). For these reason,we are mostly intereted in "FRS" which gives more general situation than the Markov property. Main purpose in this section is to show that the FRS condition leads us to countable state "sofic" shifts.(We do not need (1-3)* in this section.) There are several works on the relation between piecewise smooth dynamics and their symbolic dynamics ([6],[7] ,[17]). In particular, in case of multi-dimensional maps, piecewise linear Markov maps are studied in [6] , whose symbolic dynamics are finite state Markov shifts. These works seem to suggest a possibility of similar relation between piecewise invertible sytems with FRS and countable state sofic shifts.

Theorem 9.1 *Let $(T, X, Q = \{X_a\}_{a\in I}, \mathcal{U})$ be a piecewise invertible system with FRS. Suppose that $\mathcal{U} = \{U_0, U_1, \ldots, U_N\}$ and $X = U_0$. Then there exists a countable state sofic shift which realizes T in the following sence : Define a directed graph whose vertex set is the finite set $\mathcal{U}$, arc set is the countable alphabet I, where*

(10-1) *there is an edge c from U_i to U_j if for $\forall(a_1 \ldots a_l)$ such that $T^l X_{a_1 \ldots a_l} = U_i, X_{a_1 \ldots a_l c} \in \mathcal{L}$ and $T^{l+1} X_{a_1 \ldots a_l c} = U_j$.*

1. *This labelled graph defines a one-sided edge SFT σ with countable alphabet and a one-block map $\pi : \sigma \to \sigma'$ where σ' is a subshift with the countable alphabet I .*

2. *Let $(\sigma', \sum')$ be the one-sided sofic shift in the above ,and for $(a_0a_1\ldots) \in \sum'$ put*

$$\rho(a_0a_1\ldots) = \bigcap_{i=0}^{\infty} T^{-i}X_{a_i}.$$

Then the map $\rho : \sum' \to X$ is defined a.e. ρ is a bijective continuous conjugacy map, i.e., $T\rho = \rho\sigma'$.

3. *There exists a Markov partition for T. More specifically, let $\mathcal{V}$ be a disjoint partition generated by $\mathcal{U}$ i.e.,$\mathcal{V} = \{V_{i_0} \cap V_{i_1} \cap \ldots \cap V_{i_N} : V_{i_k} \in \{U_i, U_i^c\}\}$. Then $Q \vee \mathcal{V}$ be the Markov partition for T. In particular, we can see*

$$T(Q \vee \mathcal{V}) \subseteq \mathcal{V}.$$

We use the following facts in order to prove Theorem 10.1.

Remark P $T(T^kX_{a_1\ldots a_k} \cap X_c) = T^{k+1}X_{a_1\ldots a_k c}$.

Remark Q $(a_{-n}\ldots a_{-1})(c_0\ldots c_m)$ is admissible (i.e., $X_{a_{-n}\ldots a_{-1}c_0\ldots c_m} \in \mathcal{L}$) if and only if $\lambda(T^nX_{a_{-n}\ldots a_{-1}} \cap X_{c_0\ldots c_m}) > 0$. (It suffices to note the non-singularity of T with respect to λ.)

Next remark gives a realization of the original system $(T, X, Q, \mathcal{U})$.

Remark R Define

$$\sum^* = \{(a_0a_1\ldots) \in I^{\mathbf{N}} : \forall n \geq 0, \lambda(\bigcap_{i=0}^{n} T^{-i}X_{a_i}) > 0\}$$

and let σ^* be the shift map on $I^{\mathbf{N}}$. It follows from the non-singularity of T with respect to λ that $\sum^*$ is σ^* invariant. Put

$$\sum_{\emptyset} = \{(a_0a_1\ldots) \in \sum^* : \bigcap_{i=0}^{\infty} T^{-i}X_{a_i} = \emptyset\}.$$

The generator condition guarantees the following:

$$\text{For } \forall(a_0a_1\ldots) \in \sum^*, \ \bigcap_{i=0}^{\infty} T^{-i}X_{a_i} \text{ is at most a single point.}$$

As $\sum^* \setminus \sum_{\emptyset}$ is also σ^*- invariant, we can define $\rho : \sum^* \setminus \sum_{\emptyset} \to X$ by

$$\rho(a_0a_1\ldots) = \bigcap_{i=0}^{\infty} T^{-i}X_{a_i},$$

which is onto continuous and shift commuting. As ρ is not necessarily one to one, we have to restrict ρ to a suitable subset of $\sum^{*}\setminus\sum_{\emptyset}$ in order to obtain a one to one conjugacy map. Define

$$\sum^{*'} = \bigcap_{i=0}^{\infty} \sigma^{*^{-i}}\{(a_0a_1\ldots) \in \sum^{*}\setminus\sum_{\emptyset} : \bigcap_{i=0}^{\infty} T^{-i}X_{a_i} \in X_{a_0}\}.$$

Then we have the desired conjugacy map,i.e., $(\sum^{*'}, \sigma^{*}|_{\sum^{*'}}, \rho|_{\sum^{*'}})$ is a realization of $(T, X, Q, \mathcal{U})$.

Lemma 9.1 *$\{T^n X_{a_{-n}\ldots a_{-1}}\}_{n>0}$ is a monotone neasting sequence of subsets of X with positive Lebesgue measures, and $\bigcap_{n>0}\{T^n X_{a_{-n}\ldots a_{-1}}\}$ is exactly some $U_j \in \mathcal{U}$.*

Let us define for alphabets $a, b \in I$:

$$a \simeq b \Longleftrightarrow \text{for } \forall i, j \in \{0, 1, \ldots N\}, \text{ there is an edge from } U_i \text{ to } U_j \text{ labelled } a$$

$$\text{iff there is an edge from } U_i \text{ to } U_j \text{ labelled } b.$$

If we put $E(i, j) = \{a \in I : int(X_a \cap U_i) \neq \emptyset \text{ and } T(X_a \cap U_i) = Uj\}$, then we also can write:

$$a \simeq b \Longleftrightarrow \text{for} \forall i, j \in \{0, 1, \ldots N\}, a \in E(i, j) \text{ iff } b \in E(i, j).$$

The relation $\simeq$ defined above is an equivalence relation on the alphabet I. There are only finitely many equivalence classes $[a\ ]$ $(a \in I)$ because of the FRS condition. As mentioned in Theorem 10.1 the graph G has finitely many vertexes and countably many edges. If we replace an edge labelled a by a bundle of edges labelled $[a\ ]$, then we can obtain a quotient graph G^{*} with the same vertex set as G and with only finitely many edges. Let $\mathcal{F}(U_i)$ be the set of sequences of labels $(a_0a_1\ldots)$ of paths starting at U_i. It follows from the generator condition that each map $U_i \to \mathcal{F}(U_i)$ is bijective, so we have

$$\text{if } i \neq j \text{ then } \mathcal{F}(U_i) \neq \mathcal{F}(U_j).$$

Theorem 9.2 *The graph G gives the Fisher-cover (i.e., minimal, right-resolving, irreducible cover) of the sofic shift it defines. So the quotient graph G^{*} is also minimal.*

> **Remark S** The countable state sofic shift $(\sum', \sigma')$ can be written as a product of a finite state sofic shift and a countable state Bernoulli shift, in some case. Such examples will be given in the next section.

10 Examples and applications

In this section, we give two examples of our results in this paper which occur from number theory. Their symbolic dynamics are countable state sofic shifts and furthermore each of them is topological conjugate to a product of a finite state sofic shift and a countable state Bernoulli shift.

Example 1

Let $X = \{(x,y) : 0 \le x, y < 1\}$ and T is defined by

$T(x,y) = (-1/x - [-1/x], -y/x - [-y/x])$, where $[x] = \max\{n \in \mathbf{Z} : n \le x\}$.

Put $a(x) = -[-1/x], b(x,y) = -[-y/x]$. Since for $(x,y) \in X, (-1/x, -y/x)$ is in the following slash part (Figure 1), the index set I is given by

$$I = \{(a,b) \in \mathbf{N} \times (\mathbf{N} \cup \{0\}) : a \ge 2, a > b\}.$$

The partition of $X, Q = \{X_{(a,b)} : (a,b) \in I\}$ is defined by as follows:

$$(x,y) \in X_{(a,b)} \text{ iff } a(x) = a \text{ and } b(x,y) = b.$$

A collection of range sets $\mathcal{U}$ is consist of only two subsets of $X, U_0 = X$ and $U_1 = \{(x,y) \in X : x < y\}$. This map T is related to number theory as follows. Let us define inductively

$$a_n(x) = a(r_{n-1}(x)), b_n(x,y) = b(r_{n-1}(x), s_{n-1}(x,y)),$$

where $(r_n(x), s_n(x,y)) = T^n(x,y)(n \ge 0)$. We remark that we must restrict the domain X to the set

$$\{(x,y) \in X : T^n(x,y) \in X(\forall n > 0)\}.$$

However ,the Lebesgue measure of this set is equal to one, so we denote for the restricted set by X for convenience. From the definition of T, it is easy to see that for

$$\forall (x,y) \in X \quad x = \cfrac{1}{a_1(x) - \cfrac{1}{a_2(x) - \dots - \cfrac{1}{a_n(x) - r_n(x)}}}$$

and

$$y = \sum_{k=1}^{n} (-1)^{k-1} x x_1 x_2 \dots x_{k-1} b_k(x,y) + (-1)^n x x_1 \dots x_{n-1} s_n(x,y),$$

where

$$x_{k-1} = \cfrac{1}{a_k(x) - \cfrac{1}{a_{k+1}(x) - \dots - \cfrac{1}{a_n(x) - r_n(x)}}}.$$

T is a piecewise invertible system with FRS which satisfies all conditions of Theorem 1.1, so T has a σ-finite ergodic invariant measure. In fact, the explicit form of the invariant density $h(x,y)$ was given in [8] as follows:

$$h(x,y)=\begin{cases}\frac{2-x}{2(1-x)^2} & \text{if } x<y\\ \frac{1}{2(1-x)} & \text{if } x>y.\end{cases}$$

For convenience, let $c(x)=a(x)-b(x)$ and so inductively $c_n(x)=a_n(x)-b_n(x)$ for $n>0$. Then the admissibility rule of symbols of I is given as follow:

(A) if $c_i=1$, then $b_{i+1}\neq 0$.

Let us define a new index set

$$I'=\{(c,b):(b+c,b)\in I\}.$$

Then the admissible rule (A) can be written as follows: $(1,b)(c,0)$can not happen Let us devide I' into three subsets of I':

$$I_1'=\{(1,b):b\geq 0\}, I_2'=\{(c,b):c\geq 2, b\neq 0\}, I_3'=\{(c,0):c\geq 2\}$$

(Figure 2). Then we can obtain a quotient graph G^* with two vertexes and with finitely many bundles consists of countably many edges (Figure 3). In this example, the partition Q itself is the Markov partition .

Next we will show an example which does not satisfy the Markov condition.

Example 2

Let $X=\{(x,y):0\leq x,y<1\}$ and T is defined by

$$T(x,y)=(-\frac{1}{x}-[-\frac{1}{x}],\ \frac{y}{x}-[\frac{y}{x}]).$$

Let $a(x)=-[-1/x]$, and $b(x,y)=[y/x]$. Since for $(x,y)\in X,(-1/x,y/x)$ is in the following slash part (Figure 4), and the index set I is given by $I=\{(a,b)\in \mathbf{N}\times(\mathbf{N}\cup\{0\}):a\geq 2, a>b\neq 0\}$. The partition $Q=\{X_{(a,b)}\}_{(a,b)\in I}$ is defined by the same way as in the previous example. Let $U_0=X$ and $U_1=\{(x,y)\in X:x+y\leq 1\}$. The appearance of U_1 does not allow us to have the Markov property. However, T provides a σ-finite ergodic invariant measure with the density $h(x,y)$:

$$h(x,y)=\begin{cases}\frac{2-x}{2(1-x)^2} & \text{if } x+y<1\\ \frac{1}{2(1-x)} & \text{if } x+y>1\end{cases}$$

(See [8].) Define $c(x)=a(x)-b(x)$ and $c_n(x)=a_n(x)-b_n(x)$ inductively. Then the admissibility rule of symbols of I is given as follow: If $c_i=1$, then $c_{i+1}\neq 1$. Devide I into three groups as follows:

$$I_1=\{(a,b)\in I:c=1\}, I_2=\{(a,b)\in I:c=2\}, I_3=\{(a,b)\in I:c>2\}.$$

(Figure 5). Then we can obtain a quotient labelled graph G^* with two vertexes and finitely many bundles consists of countably many edges (Figure 6).

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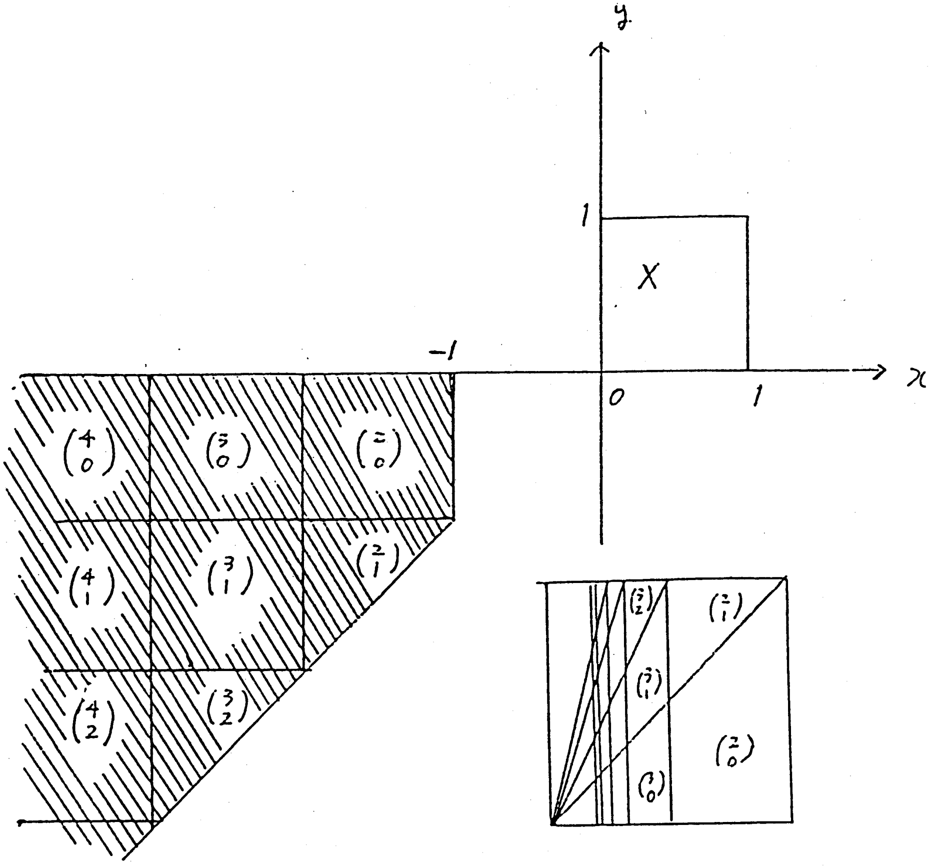
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Figure 1

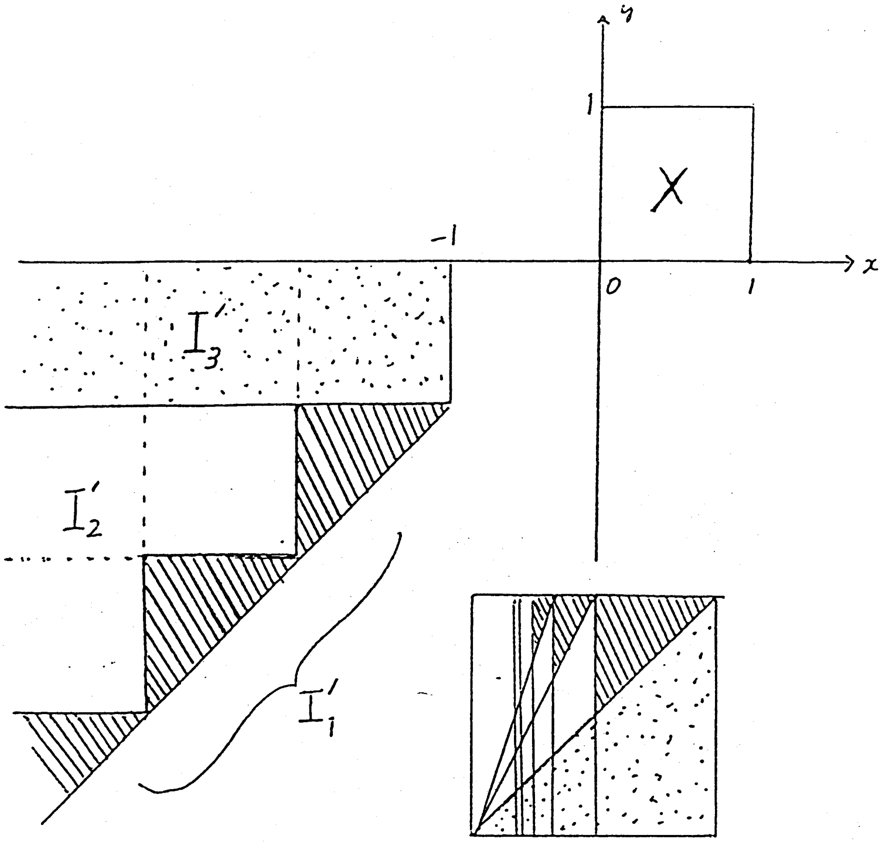
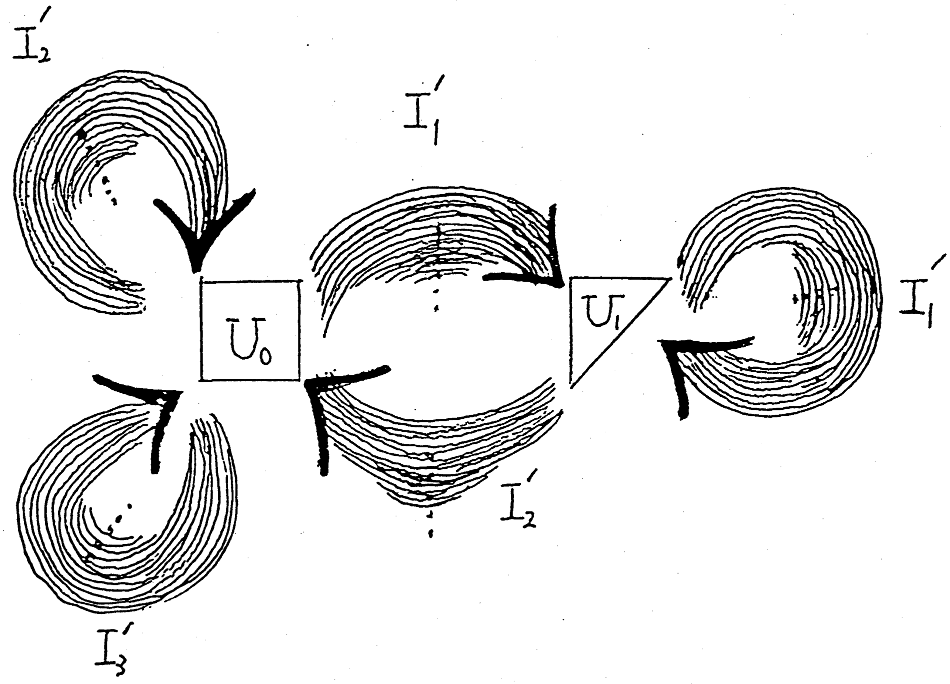


Figure 2.



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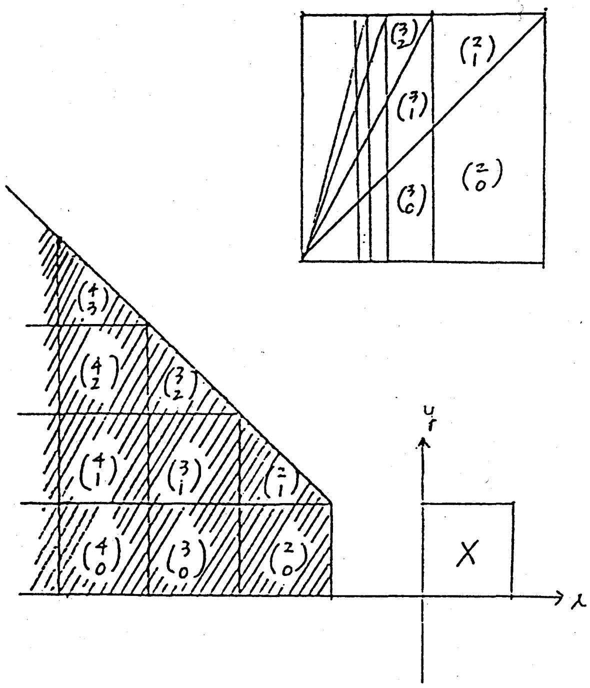
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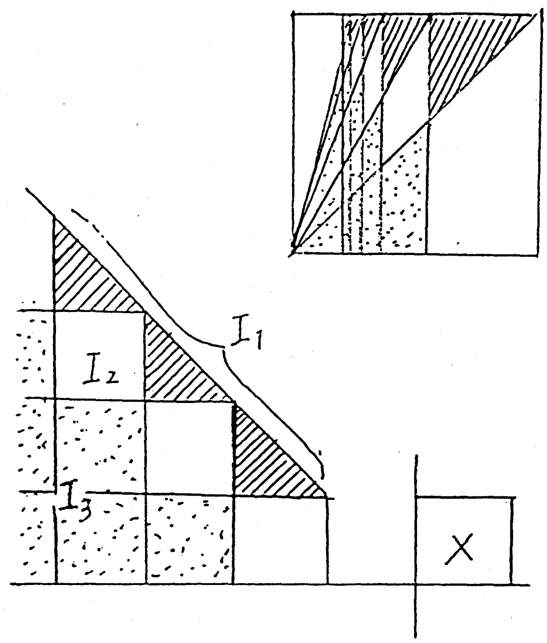

Figure 5.

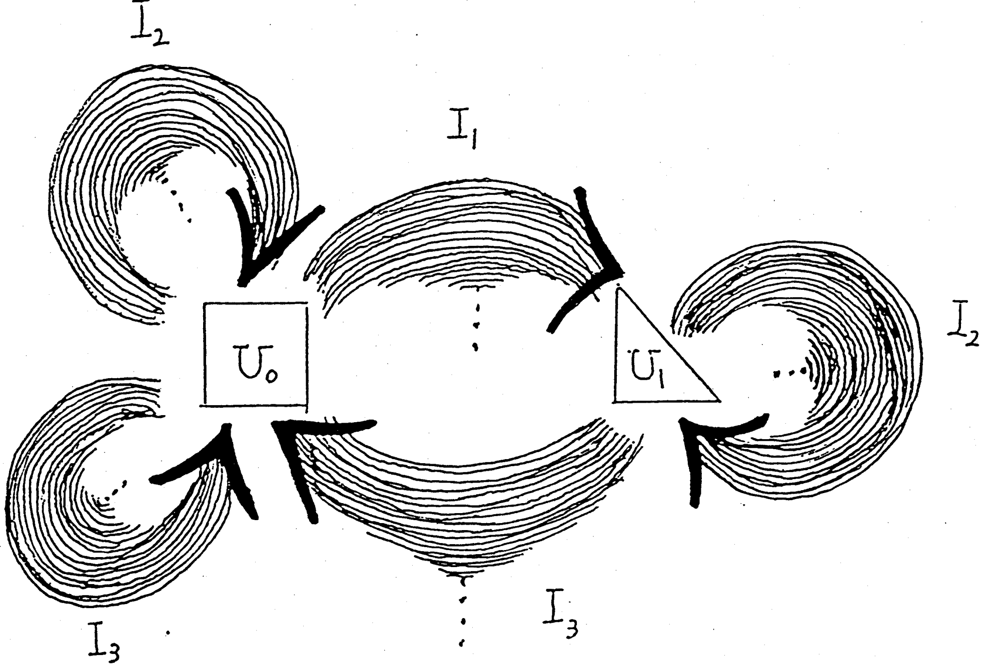

Figure 6